Dissipative transport with correlated noise

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(Received 8 July 1993; revised manuscript received 5 July 1994)

The driven-diffusion equation, proposed by Hwa and Kardar [Phys. Rev. Lett. **62**, 1813 (1989)] as a model for running sandpiles, is studied in the presence of power-correlated noise by means of the dynamical renormalization group, obtaining nonperturbative exponents. A "turbulent" regime with positive roughening exponent χ is obtained, including the Kolmogorov case $\chi=\frac{1}{3}$.

PACS number(s): 05.60.+w, 05.40.+j, 64.60.Ht

INTRODUCTION

Nonequilibrium stationary states occur in various physical contexts such as, e.g., transport processes [1–3] and growth phenomena [4]; their fluctuations have been extensively studied in recent years by means of scaling arguments, numerical simulations, and the dynamical renormalization group (DRG) (see, e.g., Ref. [5] and quotations therein). In this context, Hwa and Kardar [2] considered a noisy driven-diffusion equation as a model for running sandpiles. The driven-diffusion equation is an anisotropic extension to higher dimensions of the Burgers equation; while in the Burgers equation the field represents the velocity of the fluid, Hwa and Kardar interpreted the (one component) field as the profile of the sandpile. This interpretation is still controversial, but it turns out that the driven-diffusion equation correctly describes the density in the transport of granular materials as well as in traffic flow. The DRG treatment of Hwa and Kardar with δ-correlated noise gives a negative roughening exponent χ (corresponding to an asymptotically flat surface), with the exception of the one-dimensional case, where $\chi=0$.

In this paper, we reexamine the driven-diffusion equation both in the deterministic and in the stochastic case. In the deterministic case, one finds a strong coupling regime, which is the analog of the Kolmogorov solution [6] of the Navier-Stokes equation, with a roughening exponent $\chi=\frac{1}{3}$, independent of the dimensions. This contradicts the DRG result of Hwa and Kardar and raises the question whether the system is driven by the nonlinearity in spite of the noise. More properly, one needs to understand to what extent the strong coupling regime is influenced by the external forcing. Here we make a step along this line of reasoning and examine the effect of a forcing with long range correlation in space. The obtained infrared fixed point is characterized by exponents that depend on the nature of the noise. The roughening exponent can in fact assume positive values, including the Kolmogorov scaling, for a suitable choice of the noise. In the deterministic case, one makes use of the scale invariance of the energy transfer process, thus obtaining a uniquely defined set of exponents; in the DRG treatment, this invariance does not explicitly come into play and as a consequence one determines a range of exponents. We finally compare our results with previous work on sandslide cellular automata [7].

DETAILED CONSERVATION AND KOLMOGOROV SCALING

The driven-diffusion equation for the field $h(x_\parallel,\mathbf{x}_\perp,t)$ reads

$$\partial_t h=\nu_\parallel\partial_\parallel^2 h+\nu_\perp(\nabla_\perp)^2 h-\frac{\lambda}{2}\partial_\parallel h^2+\eta\ , \tag{1}$$

where $\eta=\eta(x_\parallel,\mathbf{x}_\perp,t)$ is the noise amplitude. Notice that anisotropy is assumed in the nonlinear term involving the derivative in the transport direction $x_\parallel$ and in the diffusion constants $\nu_\parallel,\nu_\perp$. We represent with $(\nabla_\perp)^2$ the Laplacian operator in the $(d-1)$ transverse directions $\mathbf{x}_\perp$. The field $h(x_\parallel,\mathbf{x}_\perp,t)$ describes the deviation of the surface of the sandpile from a flat incline; the role of the quadratic nonlinearity in this physical context is extensively illustrated in Ref. [2]. A discussion of the most general equation associated with the transport of a scalar quantity can be found, e.g., in Ref. [8]. The noise term is defined by its correlators:

$$\eta(\mathbf{k},\omega)=\int d^d\mathbf{x}\int dt\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\eta(\mathbf{x},t)\ ,$$
$$\langle\eta(\mathbf{k},\omega)\rangle=0\ , \tag{2}$$
$$\langle\eta(\mathbf{k},\omega)\eta(\mathbf{k}',\omega')\rangle=\mathcal{D}_0(\mathbf{k},\omega)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')(2\pi)^{d+1}\ .$$

We first examine Eq. (1) in the deterministic ($D_0=0$) case. One defines the correlation function $C(\mathbf{x},t;\mathbf{x}',t')=\langle h(\mathbf{x},t)h(\mathbf{x}',t')\rangle$, where the average is made with respect to initial conditions. If translation invariance in space holds, C satisfies the equation

$$(\partial_t+2\nu_\parallel k_\parallel^2+2\nu_\perp\mathbf{k}_\perp^2)C(\mathbf{k},t)=\int\frac{d^d\mathbf{p}}{(2\pi)^d}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})S(\mathbf{k},\mathbf{p},\mathbf{q})\ ,$$
$$C(\mathbf{k},t)\equiv C(\mathbf{k};t,t)\ , \tag{3}$$
$$S(\mathbf{k},\mathbf{p},\mathbf{q})=-\lambda k_\parallel \mathrm{Im}\langle h(\mathbf{k},t)h(\mathbf{p},t)h(\mathbf{q},t)\rangle\ ,$$

Im(x) being the imaginary part of x. One immediately verifies the detailed conservation

$$S(\mathbf{k},\mathbf{p},\mathbf{q})+S(\mathbf{p},\mathbf{q},\mathbf{k})+S(\mathbf{q},\mathbf{k},\mathbf{p})=0\ ,\quad (k_\parallel+p_\parallel+q_\parallel=0)\ .$$

As a consequence, upon integrating over the $\mathbf{k}$ variable, one obtains the total conservation law,

$$\partial_t \int d^d\mathbf{k}\, C(\mathbf{k},t)+\int d^d\mathbf{k}(2\nu_\parallel k_\parallel^2+2\nu_\perp \mathbf{k}_\perp^2)C(\mathbf{k},t)=0\ , \tag{4}$$

which exhibits the conservative nature of the nonlinear term. It is then natural to define an "energy density" $H(t)=\frac{1}{2}\int[d^d\mathbf{k}/(2\pi)^d]C(\mathbf{k},t)=\frac{1}{2}\langle h^2(\mathbf{x},t)\rangle$. In the inertial regime, where $\nu_\parallel=\nu_\perp=0$, we have $d/dtH(t)=0$.

In order to discuss the behavior in the transport direction, we average over the transverse directions by defining the spectral density $H(k_\parallel,t)$:

$$H(t)=\int_0^{+\infty}dk_\parallel H(k_\parallel,t)=\int_0^{+\infty}dk_\parallel\int_0^{+\infty}dk_\perp H(k_\parallel,k_\perp,t)\ ,$$

which satisfies the equation

$$\begin{aligned}
&\partial_t H(k_\parallel,t)+2\nu_\parallel k_\parallel^2 H(k_\parallel,t)+2\nu_\perp\int_0^{+\infty}dk_\perp k_\perp^2 H(k_\parallel,k_\perp,t)=T(k_\parallel,t)\ ,\\
&T(k_\parallel,t)=\frac{S_{d-1}}{(2\pi)^d}\int_0^{+\infty}dk_\perp k_\perp^{d-2}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})S(\mathbf{k},\mathbf{p},\mathbf{q})\ ,
\end{aligned}\tag{5}$$

where S_d is the surface area of the d-dimensional unit sphere. The nonlinearity drives the excitation towards higher momenta, indeed in the inertial regime the velocity is proportional to the field amplitude [see Eq. (1)]. Hence the energy flux through momentum $k_\parallel$ is given by $\Pi(k_\parallel)=-\int_0^{k_\parallel}dk'_\parallel T(k'_\parallel)$, where the conservation law $\int_0^{+\infty}dk'T(k')=0$ has been used. In the inertial regime, a scale-invariant energy flux $\Pi(k_\parallel)$ is compatible with Eq. (5). In fact, if a constant energy flux ϵ [9] is injected at momentum $(k_\parallel)_i$, at steady state one has

$$T(k_\parallel)+\epsilon\delta(k_\parallel-(k_\parallel)_i)=0\ , \tag{5'}$$

which implies $\Pi(k_\parallel)=\epsilon$ $(k_\parallel>(k_\parallel)_i)$. By dimensional analysis, one fines the Kolmogorov spectrum

$$H(k_\parallel)\sim\epsilon^{2/3}k_\parallel^{-5/3}\ ,\quad k_\parallel>(k_\parallel)_i\ . \tag{6}$$

Naive scaling in terms of the roughening exponent χ and of the dynamical exponent z $(h\sim x_\parallel^\chi, t\sim x_\parallel^z)$ leads to the same conclusion $(\chi=\frac{1}{3},\ z=\frac{2}{3})$ if one assumes two scale invariances: (i) the scale invariance of the equation in the inertial regime, implying $\chi+z=1$; (ii) the scale invariance of the energy transfer, implying $z=2\chi$.

This stationary regime describes a cascade toward higher momenta up to the point at which viscosity starts competing with the nonlinear term. We stress that the DRG result of Hwa and Kardar fails in recovering this "turbulent" behavior; in particular, the roughening exponent χ is always negative or at the most equal to zero in one dimension.

RENORMALIZATION GROUP RESULTS WITH POWER NOISE

We reexamine the DRG analysis by assuming a noise with long range correlation in space. As long as the nonlinearity involves the single component $k_\parallel$, singular behavior is expected in the direction of transport. It is then natural to assume an anisotropic noise. Power-correlated noise has been considered for the Navier-Stokes equation by De Dominicis and Martin [10] and by Yakhot and Orszag [11]. We have

$$D_0(\mathbf{k},\omega)=2D_0|k_\parallel|^{-y}\ ,$$

where y is arbitrary. The basic formalism introduced here essentially follows the lines of various previous papers (in particular, Refs. [2], [11], [12], and [13]) so that we will merely give the results. In terms of the unperturbed propagator $G_0(\mathbf{k},\omega)$, Eq. (1) has the form

$$\begin{aligned}
&h(\mathbf{k},\omega)=-\frac{\lambda}{2}ik_\parallel G_0(\mathbf{k},\omega)\int\frac{d^d\mathbf{q}}{(2\pi)^d}\int\frac{d\mu}{2\pi}h(\mathbf{q},\mu)h(\mathbf{k}-\mathbf{q},\omega-\mu)+G_0(\mathbf{k},\omega)\eta(\mathbf{k},\omega)\ ,\\
&G_0(\mathbf{k},\omega)=(-i\omega+\nu_\parallel k_\parallel^2+\nu_\perp\mathbf{k}_\perp^2)^{-1}\ .
\end{aligned}\tag{7}$$

We study the hydrodynamic regime $\mathbf{k},\omega\to 0$. No divergencies are generated by ω and $\mathbf{q}_\perp$ integrations, which are performed over $(-\infty,+\infty)$ and $(0,\Lambda_\perp)$. The renormalized functions $G^{(R)}$ and $D^{(R)}$, valid for small momenta $k_\parallel^<$ $(k_\parallel^< < e^{-l}\Lambda_\parallel)$, are obtained by integrating over large $q_\parallel^>$ $(e^{-l}\Lambda_\parallel<q_\parallel^> <\Lambda_\parallel)$. We represent in Fig. 1 the exact propagator G, defined by $h(\mathbf{k},\omega)=G(\mathbf{k},\omega)\eta(\mathbf{k},\omega)$, and the exact noise correlation function $\mathcal{D}$ defined by

$$\langle h(\mathbf{k},\omega)h(\mathbf{k}',\omega')\rangle=G(\mathbf{k},\omega)G(\mathbf{k}',\omega')\mathcal{D}(\mathbf{k},\omega)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')(2\pi)^{d+1}\ .$$

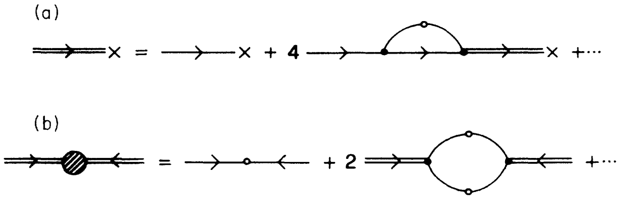

FIG. 1. (a) Equation for the field h applied to a source η. The double arrow represents the exact propagator G, the single arrow represents the bare propagator G_0, the open circle represents the zeroth order average over the noise source, and the black dot represents the vertex $-(\lambda/2)\iota k_\parallel \int d^d\mathbf{q}/(2\pi)^d \int d\mu/2\pi$. (b) Equation for the noise correlation function $\mathcal{D}(\mathbf{k},w)$.

We further notice that, as observed in Ref. [12], the invariance of Eq. (1) under the "Galilean" transformation,

$$x_\parallel \to x_\parallel - \lambda \tilde{h} t \ , \quad h \to h + \tilde{h} \ , \tag{8}$$

implies that the coupling constant λ does not renormalize. The remaining parameters up to second order in λ transform according to

$$\begin{aligned} \nu_\perp^R &= \nu_\perp \ , \\ \nu_\parallel^R &= \nu_\parallel \left[1 + A_d \bar{\lambda}^2 \frac{e^{\epsilon l} - 1}{\epsilon} \right] \ , \\ D^R &= D_0 \ , \quad y > -2 \ , \\ D^R &= D_0 \left[1 + \tfrac{1}{2} N_d \bar{\lambda}^2 \frac{e^{\epsilon l} - 1}{\epsilon} \right] \ , \quad y = -2 \ , \end{aligned} \tag{9}$$

where

$$\bar{\lambda}^2 = \frac{D_0 \lambda^2}{\nu_\parallel^3} \left[\frac{\nu_\parallel}{\nu_\perp} \right]^{(d-1)/2} \frac{1}{\Lambda_\parallel^\epsilon} \ ,$$

$$A_d = \frac{S_{d-1}}{(2\pi)^d} \int_0^{\Lambda_\perp} dp \frac{p^{d-2}}{(1+p^2)^2} \left[1 - \frac{1}{1+p^2} \right] \ ,$$

$$N_d = \frac{S_{d-1}}{(2\pi)^d} \int_0^{\Lambda_\perp} dp \frac{p^{d-2}}{(1+p^2)^3} \ .$$

From Eq. (9), one can determine the renormalization group (RG) equations by rescaling $k_\parallel$, $k_\parallel \to k'_\parallel = k_\parallel e^l$, with the following definitions for the exponents:

$$\mathbf{k}'_\perp = \mathbf{k}_\perp e^{\zeta l} \ , \quad \omega' = \omega e^{zl} \ , \quad h'(\mathbf{k}',\omega') = h(\mathbf{k},\omega) e^{-wl} \ . \tag{10}$$

The roughening exponent $\chi, h'(\mathbf{x}',t') = h(\mathbf{x},t) e^{-\chi l}$, is then $\chi = w - z - (d-1)\zeta - 1$.

In the case $y > -2$, the RG equations are

$$\begin{aligned} \frac{d\nu_\parallel(l)}{dl} &= \nu_\parallel(l)[z - 2 + A_d \bar{\lambda}^2(l)] \ , \\ \frac{d\nu_\perp(l)}{dl} &= \nu_\perp(l)[z - 2\zeta] \ , \\ \frac{dD(l)}{dl} &= D(l)[3z + (d-1)\zeta + y + 1 - 2w] \ , \\ \frac{d\bar{\lambda}(l)}{dl} &= \tfrac{1}{2}\bar{\lambda}(z) \left[\epsilon - \frac{7-d}{2} A_d \bar{\lambda}^2(l) \right] \ , \end{aligned} \tag{11}$$

with a nontrivial stable fixed point $\bar{\lambda}^* = [2\epsilon/(7-d)A_d]^{1/2}$ under the condition $y < 3$, or $y \ge 3$, $d < 7$. Notice that as long as the canonical dimensions of the field h increase with y, $([h]_0 = (2-d+y)/2)$, one is allowed to consider, among all the nonlinear terms compatible with Galilean invariance, the single term $h \partial_\parallel h$, provided only that the inequality $[h \partial_\parallel h]_0 = 1 + y - d < 0$ holds. Taking into account that λ and $\nu_\perp$ do not renormalize to all orders and, furthermore, that if $y > -2$, D_0 also does not renormalize, and one easily concludes that the exponents χ, z, ζ can be determined exactly. As a result, from Eq. (11) one has

$$\begin{aligned} \chi &= \frac{1-d+2y}{7-d} \ , \\ z &= \frac{6-2y}{7-d} \ , \\ \zeta &= \frac{z}{2} \ . \end{aligned} \tag{12}$$

In (12), the last relation implies normal diffusion in the transverse directions $(x_\perp \approx t^{\zeta/z})$. One also verifies the scaling relation $\chi + z - 1 = 0$. As previously observed, this is the condition under which the nonlinear term of Eq. (1) allows for scale-invariant solutions; in the case of rough $(\chi > 0)$ surfaces, it implies a transport faster than ballistic: $x_\parallel \approx t^{1/z}$, $z < 1$. The energy spectrum scales as $H(k_\parallel) \sim k_\parallel^{-(2\chi+1)}$; in particular, the Kolmogorov exponent $-\frac{5}{3}$ is recovered if $y = (d+2)/3$, so that $z = \frac{2}{3}$, $\zeta = \chi = \frac{1}{3}$ independent of the dimension d; in this case, the field has Gaussian fluctuations in time $(h \approx t^{\chi/z} = t^{1/2})$. With conservative noise $(y = -2)$, the last two equations in (11) become

$$\begin{aligned} \frac{dD(l)}{dl} &= D(l)[3z + (d-1)\zeta + y + 1 - 2w + \tfrac{1}{2} N_d \bar{\lambda}^2(l)] \ , \\ \frac{d\bar{\lambda}(l)}{dl} &= \tfrac{1}{2}\bar{\lambda}(l)\{\varepsilon + \tfrac{1}{2}\bar{\lambda}^2(l)[N_d - (7-d)A_d]\} \ . \end{aligned} \tag{13}$$

Here, if $d \ge 2$, one has the trivial fixed point $\bar{\lambda}^* = 0$ with exponents $\chi = -d/2$, $z = 2$, $\zeta = 1$. In the region $d < 2$, one finds instead a fixed point $\bar{\lambda}^* = (32\pi\epsilon)^{1/2}$ with $\chi = 1 - d$, $z = d$, $\zeta = d/2$; this corresponds, in $d = 1$, to ballistic diffusion $(\chi = 0, z = 1)$. Finally, when $y < -2$, one notices that the leading contribution to the noise correlation still goes as $k_\parallel^2$, so that the same behavior as that for $y = -2$ is expected. For consistency, we checked the results obtained in the limit $d \to 1$ by directly renormalizing the $d = 1$ Burgers equation. If $y > -2$, we obtain consistent results, but for $y \leq -2$ the trivial fixed point is unstable, a sign of the failure of the perturbative approach.

Let us summarize the main results. One finds a positive roughening exponent χ under the condition $(d-1)/2 \leq y < \min(3,d-1)$, $(0 \leq \chi < 1)$, the diffusion here being faster than ballistic $(z \leq 1)$; the Kolmogorov regime is found in this range under the condition $d > \frac{5}{2}$. The case $y=3$ corresponds to $\chi=1$, the threshold above which the surface is unstable. The exponent χ is negative if $y < \min[(d-1)/2,3]$. It is interesting to notice that in this case the behavior is always superdiffusive $(1 < z < 2)$. We add some comments on the time dependence of the output current $J(t)$ [2] through the edge of the system $x_{\parallel} = L_{\parallel}$. In the infrared limit, one has

$$J(t) \approx (\lambda/2) \int d^{d-1}\mathbf{x}_{\perp} h^2(x_{\parallel}, \mathbf{x}_{\perp}, t) ,$$

where

$$\langle J(t)J(0) \rangle \approx L_{\perp}^{d-1} t[4\chi + (d-1)\zeta]/z , \tag{14}$$

where $L_{\perp}$ is the transverse size of the system. From (14), one obtains the frequency spectrum $S_J(f)$:

$$S_J(f) = \frac{1}{2\pi} \int dt\, e^{-ift} \langle J(t)J(0) \rangle \approx \frac{1}{f^{\phi_J}} , \quad \phi_J = \frac{1+y}{z} . \tag{15}$$

Similarly, the spectrum of the dissipated energy $E(t) \approx (\lambda/2) \int d^d\mathbf{x} h^2(\mathbf{x},t)$ is

$$S_E(f) \approx \frac{1}{f^{\phi_E}} , \quad \phi_E = \frac{2+y}{z} . \tag{16}$$

These estimates for ϕ_J and ϕ_E are correct, provided that $J(t)$ and $E(t)$ have correlators decaying in time [14].

CONCLUSIONS

In this paper, we examined the scaling in transport processes described by the driven-diffusion equation. We focused on the consistency of the DRG results with the steady state solution of the deterministic problem. This solution, describing a scale independent energy flux in the transport direction, is proper for the strong coupling regime and gives a positive roughening exponent. Surprisingly enough, it is unstable upon addition of δ-correlated noise, or at least this is the conclusion of DRG analysis (notice that in this case, the DRG result for the exponents is correct to all orders of the perturbation theory). Only by assuming a power-correlated noise does one recover the Kolmogorov solution or more generally have a positive roughening; the result is always explicitly dependent on the noise [see Eq. (12)]. The naive scaling of the equation is consistent with the exponents (12), but fails when $y \leq -2$. A direct numerical analysis of the one-dimensional 1D driven-diffusion equation with δ-correlated noise [15] has been recently performed, obtaining full agreement with the DRG results of Hwa and Kardar. We expect that the exact exponents obtained here should also be confirmed. Interestingly enough, the DRG results from the Kardar-Parisi-Zhang equation with spatial correlations have been recovered in simulations of correlated surface growth [16].

Scaling behavior is found in sandpile cellular automata, where the height or the slope of the sandpile, defined over the lattice, propagates according to given dynamical rules when it exceeds a threshold value. Particles are added to randomly chosen sites and leave the system from a given boundary. It has been shown [7,17] that a class of this automata can be described in the hydrodynamic limit by a singular diffusion equation,

$$\frac{\partial S}{\partial t} = \nabla \cdot [D(S) \nabla S] , \quad D(S) = S^{-\phi} ,$$

with boundary conditions consistent with the excitation mechanism. The exponent ϕ is model dependent and determines, together with the dimension d and the excitation, the scaling of S with the system size $S \sim L^{-\beta}$. One gets the relation $z = 2 - \beta\phi$, the counterpart of $\chi + z = 1$ from the driven-diffusion equation, and is tempted to make a connection between the two. In the case of the two-state model $(d=1, \phi=3, \beta=\frac{1}{2})$, upon identifying S with the slope of h, $[S]=[\partial h/\partial x]$, one indeed obtains the correct numbers $z=\frac{1}{2}$, $\chi=\frac{1}{2}$, but this seems a mere coincidence, as it is not confirmed in other cases.

ACKNOWLEDGMENTS

We thank F. Nori for introducing us to self-organized criticality and M. Kardar for reading the manuscript and making useful comments.

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